Financial Modelling with Jump Processes

1 Instructions

You will have to write a report in English. There will be a defense in English too. The defense date is not fixed yet. One week before, I expect a printed version of your dissertation. The code should be included too. Furthermore you will send me your .o, .h, .c, .r, .exe...

I will particularly pay attention to the quality of the dissertation and the quality of algorithms. You may code in any language you want. However,I strongly recommend C#, C ++ for the algorithms plus an interface in R/Matlab/Excel for the statistic analysis.

You will most of the time work on your own. The next course (and the last) will take place on the Friday, the 30 rd of November. Don't hesitate in between to contact me at amel.bentata@gmail.com.

Good luck !

Ps : Pour les négociations, adressez-vous à Adrien qui me tranmettra ! Pour les coquilles adressez-vous à Lionel ou Alexandre qui se feront un plaisir de m'en parler !

2 Black & Scholes in a multivariate framework (Club Med...)

In the sequel, we consider a market described by a scenario space $(\Omega, \mathcal{F})$, asset prices $(S_t)_{t\in[0,T]}$ and an information flow $(\mathcal{F}_t)_{t\in[0,T]}$, which is the history of the assets. In all examples, the numeraire is $S_t^0 = \exp(rt)$; the discount factor is then given by $B(t,T) = \exp(-r(T-t))$.

Consider a multivariate model with d assets

$$S_T^i = S_0^i + r\,S_t^i dt + \int_0^T S_t \sigma^i dW_t^i, \tag{1}$$

where σ^i is a constant taking values in $\mathbb{R}^+$ representing the volatility of asset i, W is a d-dimensional Wiener process The Wiener processes W^i are correlated

$$\forall 1 \leq (i,j) \leq d, \langle W^i, W^j \rangle_t = \rho_{i,j} t,$$

with $\rho_{ij} > 0$ and $\rho_{ii} = 1$. An index is defined as a weighted sum of asset prices

$$I_t = \sum_{i=1}^d w_i S_t^i, \quad w_i > 0, \quad \sum_1^d w_i = 1, \quad d \geq 2.$$

The value $C_{t_0}(T,K)$ at time t_0 of an index call option with expiry $T > t_0$ and strike $K > 0$ is given by

$$C_{t_0}(T,K) = e^{-r(T-t_0)} E^{\mathbb{P}}[\max(I_T - K, 0) | \mathcal{F}_{t_0}]. \tag{2}$$

Let us consider the case when the Wiener processes W^i are homogeneously correlated, implying that the correlation matrix of the Wiener processes, denoted

Σ^W is of the form

$$\Sigma^W = \begin{pmatrix} 1 & \cdots & \rho_W \\ \vdots & \ddots & \vdots \\ \rho_W & \cdots & 1 \end{pmatrix}. \tag{3}$$

We choose the index to be equally weighted :

$$\forall 1 \leq i \leq d \quad w_i \equiv \frac{1}{d}.$$

1. How would you simulate a possible trajectory of two Brownian motions correlated to ρ ($\rho \in [0,1]$) ?
2. We put $t_0 = 0$ in the sequel. Let us now choose a basket of $d = 30$ assets, following the model (1) and specified by the parameters,

$$S_0^i = 100, \quad r = 0.04, \quad \rho_W = 0.1, \quad . \tag{4}$$

   We generate the σ^i's (per annum), uniformly via

$$\sigma^i \sim \mathcal{U}_{[0,1;0,2]}. \tag{5}$$

   For a maturity $T = 10/252$ and strikes K (out of the money),

$$K = 101,\ 102,\ 103 \cdots,\ 115,\ 116,\ 117,$$

   compute $\hat{C}_0(T,K)$ using a Monte Carlo method with $N = 10^5$ trajectories. Plot these prices with respect to the logmoneyness.
3. Let $\hat{\sigma}(T,K)$ be the standard deviation of $\hat{C}_0(T,K)$, the Monte Carlo estimator. Propose a 90%-confidence interval for $C_0(T,K)$.
4. What is the complexity of your algorithm ? Conclusion.
5. What is the Itô decomposition of I_t ? What is its volatility ? We will denote it σ_t^I.
6. For short maturities ($T \sim 0$), we will assume that

$$\forall 1 \leq i \leq d \quad S_t^i \sim S_0^i.$$

   Deduce that one can assimilate (I_t) to a Black-Sholes model with a certain (deterministic) volatility.
7. Propose two approaches, one numerical and the other analytical to approximate the $C_0(T,K)$'s. What are their complexities ?
8. Conclusion.
9. At the money case : we still consider short maturities and I stipulate that

$$\frac{1}{\sqrt{T}} C(T,K) \sim_{T \sim 0} g(I_0, \sigma_0^I)$$

   Propose a numerical method to identify such a function g which "match" your option prices. Conclusion.
10. Empirical study : what about the CAC40 ?

3 Simulation of a jump-diffusion process

1. Propose an algorithm that simulates a possible trajectory of a Poisson process $(N_t)$with intensity $\lambda = 2$ on the time interval $[0, 1]$. Plot one trajectory.
2. On the time interval $[0, T]$, we would like to compute the mean number of jumps that occurs. Clearly, this mean should be of the form

   $$m \equiv m(T, \lambda)$$

   Propose a numerical method to specify the function m.
3. We recall that a compound Poisson process with intensity $\lambda > 0$ and jump size distribution f is a stochastic process (X_t) defined as,

   $$X_t = \sum_{i=1}^{N_t} Y_i, \tag{6}$$

   where jumps sizes are independent and identically distributed with distribution f and (N_t) is a Poisson process with intensity λ independent from $(Y_i)_{i\geq 1}$.

   Example : Assume that the jumps are gaussian distributed. Propose an algorithm to simulate (X_t) on a time interval $[0, T]$.
4. Plot a trajectory of X_t on the time interval $[0, 1]$ with the parameters :

   $$\lambda = 2,$$
   $$\mathbb{E}[Y_1] = -0.05, \mathbb{V}ar(Y_1) = 0.1.$$

5. An improved algorithm for compound Poisson process.

   Sums of i.i.d exponential variables bear a close relationship with the order statistics of uniformly distributed random variables. Namely, we already know that the number of jumps N_t of a compond Poisson process on the interval $[0, T]$ is a Poisson random variable with parameter λT. Furthermore, conditionally on N_T, the exact moments of jumps on this interval have the same distribution as N_T independent random numbers, uniformly distributed on this interval, rearranged in increasing order.

   Propose an improved algorithm for the simulation of a compound Poisson process. In which sense is it better ?
6. We recall that a jump-diffusion process is a stochastic process which presents a Gaussian component and a jump component of Compound Poisson type. Both component are independent. Namely, the dynanmic of such a stochastic process is

   $$X_t = X_0 + b\,t + \sigma W_t + \sum_{i=1}^{N_t} Y_i. \tag{7}$$

   Propose an algorithm that simulates X_t on a fixed time grid $(t_1, t_2, \cdots, t_n)$.
7. What is the complexity of this algorithm ?
8. With the same parameters as before plus $\sigma = 0.1$ and $b = 1$, simulate a trajectory of (X_t) on the time grid $(0; 1/252; \cdots; 10/252)$ and plot it.

4 Pricing European call under the Merton model

The first application of jump processes in option pricing was introduced by R. Merton. He considered the jump-diffusion model whose dynamic under $\mathbb{P}$ is :

$$S_t = S_0 \, \exp\left(\mu \, t + \sigma \, W_t + \sum_{i=1}^{N_t} Y_i\right),$$

where (W_t) denotes a standard Wiener process, (N_t) is a Poisson process with intensity λ independent from (W_t) and the Y_i's are Gaussian random variables i.i.d and independent from (W_t) and (N_t). There are many possible choices for a risk-neutral measure but we propose the one obtained by changing the drift but leaving the other component unchanged. Basically, μ is chosen such that the discounted asset price $(e^{-rt} \, S_t)$ is a martingale under the risk neutral measure $\mathbb{Q}$:

$$\text{Under } \mathbb{Q} \quad S_t = S_0 \, \exp\left(\tilde{\mu} \, t + \sigma \, \tilde{W}_t + \sum_{i=1}^{N_t} Y_i\right),$$

where $(\tilde{W}_t)$ denotes a standard Wiener process, (N_t) and (Y_i) are as above and independent from $(\tilde{W}_t)$ and $\tilde{\mu}$ is chosen such that $e^{-rt} \, S_t$ is a martingale under $\mathbb{Q}$.

1. Prove that
$$\tilde{\mu} = r - \frac{1}{2} \, \sigma^2 - \lambda \, \mathbb{E}[e^{Y_1} - 1].$$
2. Is (S_t) a Markovian process ?
3. Deduce that one can express the European call price
$$C_t(T, K) = e^{-r(T-t)} \, \mathbb{E}\left[(S_T - K)^+ | \mathcal{F}_t\right] \quad \forall T \geq t,$$
as a weighted sum of Black-Scholes prices C^{BS}.
Hint : Condition on the number of jumps N_t.
4. Deduce an analytical approximation of $C_t(T, K)$ and an algorithm which compute this for any $T \geq t$ and any K.
5. What is the complexity of this algorithm ?
6. Propose another method, then its algorithm to compute the $C_t(T, K)$'s for any $T \geq t$ and any K.
7. Compare their complexities.
8. Numerical application : we propose this set of values
$$S_0 = 100, t = 0, T = 10/252, \sigma = 0.1$$
$$r = 0.04, \quad \lambda = 2, K = 101, \ 102, \ 103 \cdots, \ 115, \ 116, \ 117,$$
$$\mathbb{E}[Y_1] = -0.05, \mathbb{V}ar(Y_1) = 0.1.$$
With a graph, propose a qualitative comparison between this two methods and a simple confidence interval. Conclusion ?

9. Short maturity asymptotic for call option prices in the Merton model : Let $\psi(z)$ be the double exponential tail of the density $p(dx)$ of a Gaussian distribution :

$$\psi(z) = \int_z^{+\infty} dx\ e^x \int_x^{\infty} p(du), \quad z > 0. \tag{8}$$

I claim that :
- ATM : $C_0(T,K)/\sqrt{T}$ is (close to) a simple function of the parameters of the model when $T \sim 0$;
- OTM : $C_0(T,K)/T$ is (close to) a simple function of the parameters of the model when $T \sim 0$.

Propose a simple qualitative method which helps us to identify which parameters are involved ATM and OTM and how is this function like.

10. Conclusion : what is the interest of this project ? What did you understand about the complexity of stochastic modelling in the pricing area ?